Newton Polygons of L-Functions

Phong Le

Department of Mathematics
University of California, Irvine

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Laurent Polynomials

Let $q = p^a$ where p is a prime and a is a positive integer. Let $\mathbb{F}_q$ denote the field of q elements.
For a Laurent polynomial $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we may represent f as:

$$f = \sum_{j=1}^{J} a_j x^{V_j}, a_j \neq 0,$$

where each exponent $V_j = (v_{1j}, \ldots, v_{nj})$ is a lattice point in $\mathbb{Z}^n$ and the power x^{V_j} is the product $x_1^{v_{1j}} \cdot \ldots \cdot x_n^{v_{nj}}$.

Example

$$\begin{array}{ccccccc} f(x_1, x_2) & = & \frac{2}{x_1} & + & 10x_1x_2^2 & + & 82 \\ \text{lattice points} & = & \{(-1, 0) & , & (1, 2) & , & (0, 0)\} \end{array}$$

$\mathbb{F}_p(\Delta)$

Let $\Delta(f)$ denote Newton polyhedron of f, that is, the convex closure of the origin and $\{V_1, \ldots, V_J\}$, the integral exponents of f.

Definition

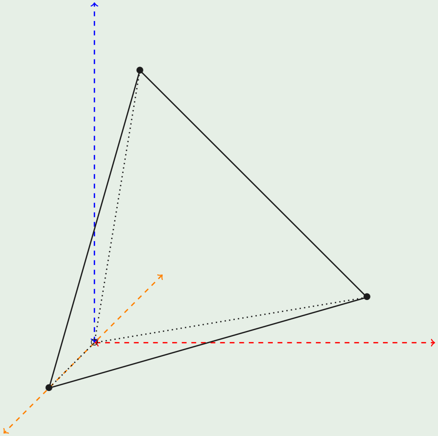
Given a convex integral polytope Δ which contains the origin, let $\mathbb{F}_q(\Delta)$ be the space of functions generated by the monomials in Δ with coefficients in the algebraic closure of $\mathbb{F}_q$, a field of q elements.

In other words,

$$\mathbb{F}_q(\Delta) = \{f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \Delta(f) \subseteq \Delta\}.$$

The polytope Δ

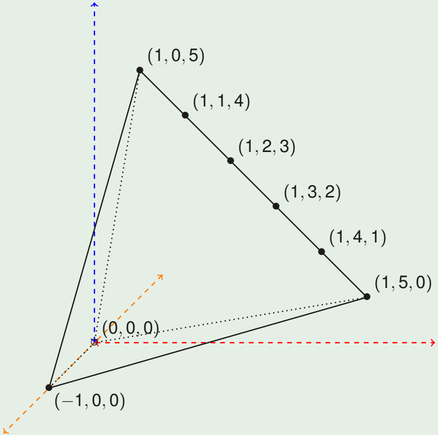
Example



Let Δ be the polytope generated by $f(x, y, z) = 1/z + x^5 z + y^5 z$.

The polytope Δ

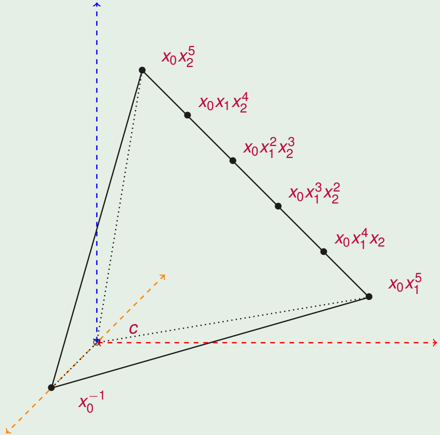
Example



It is also the convex closure of the lattice points (including interior points).

The polytope Δ

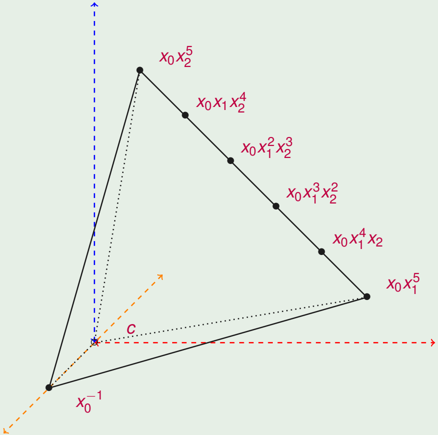
Example



We can correspond each lattice point to a monomial in n variables (including interior points).

The polytope Δ

Example



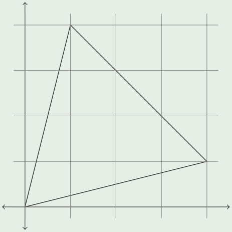
$\mathbb{F}_p(\Delta)$ is space of functions the generated by these monomials (including interior points).

Facial Restriction

Let δ be a face of Δ of arbitrary dimension. Define f_δ to be the restriction of f to the terms with exponent in δ:

$$f_\delta = \sum_{V_j \in \delta} a_j x^{V_j}.$$

Example ($f = xy^4 + x^4y$)



There are three faces of Δ away from the origin.

Facial Restriction

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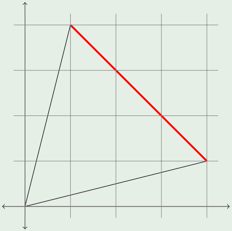

There is one face δ_1 of codimension 1.
$f_{\delta_1} = f = xy^4 + x^4 y$.

Facial Restriction

Let δ be a face of Δ of arbitrary dimension. Define f_δ to be the restriction of f to the terms with exponent in δ:

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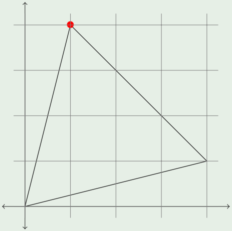


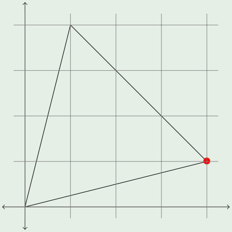
There are two faces of codimension 2. One is face δ_2 which consists of the point $(1,4)$. Hence $f_{\delta_2} = xy^4$

Facial Restriction

Let δ be a face of Δ of arbitrary dimension. Define f_δ to be the restriction of f to the terms with exponent in δ:

$$f_\delta = \sum_{V_j \in \delta} a_j x^{V_j}.$$

Example ($f = xy^4 + x^4y$)



Similarly the other face δ_3 consists of $(4, 1)$. Hence $f_{\delta_3} = x^4 y$.

$M_q(\Delta)$

Definition

The Laurent polynomial f is called non-degenerate if for each closed face δ of $\Delta(f)$ of arbitrary dimension which does not contain the origin, the n partial derivatives

$$\{\frac{\partial f_\delta}{\partial x_1}, \ldots, \frac{\partial f_\delta}{\partial x_n}\}$$

have no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_q$.

Definition

Let $M_q(\Delta)$ be the functions in $\mathbb{F}_q(\Delta)$ that are non-degenerate.

Definition of the L-function

Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let ζ_p be a p-th root of unity and $q = p^a$. For each positive integer k, consider the exponential sum:

$$S_k^*(f) = \sum_{(x_1,\ldots,x_n)\in\mathbb{F}_{q^k}^*} \zeta_p^{Tr_k f(x_1,\ldots,x_n)}.$$

The behavior of $S_k^*(f)$ as k increases is difficult to understand.

To better understand $S_k^*(f)$ we define the L-function as follows:

$$\begin{array}{ccccc} \mathbb{F}_q, & \mathbb{F}_{q^2}, & \cdots & \mathbb{F}_{q^k}, & \cdots \\ S_1^*(f), & S_2^*(f), & \cdots & S_k^*(f), & \cdots \\ S_1^*(f)T+ & S_2^*(f)\frac{T^2}{2}+ & \cdots+ & S_k^*(f)\frac{T^k}{k}+ & \cdots \end{array}$$

$$L^*(f,T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f)\frac{T^k}{k}\right).$$

By a theorem of Dwork-Bombieri-Grothendieck $L(f,T)$ is a rational function.

$NP(f)$

Adolphson and Sperber showed that if f is non-degenerate

$$L^*(f,T)^{(-1)^{n-1}} = \sum_{i=0}^{\infty} A_i(f)T^i, \quad A_i(f) \in \mathbb{Z}[\zeta_p]$$

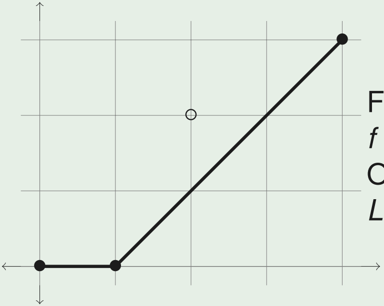
is a polynomial of degree $n!\,Vol(\Delta)$.

Definition

Define the Newton polygon of f, denoted $NP(f)$ to be the lower convex closure in $\mathbb{R}^2$ of the points

$$(k, \mathrm{ord}_q A_k(f)), k = 0, 1, \ldots, n!\,Vol(\Delta).$$

Example



For $p = q = 3$ and
$f = \frac{1}{x_1} + x_1 x_2^2 + x_1 x_3^2$.
One can computed directly:
$L(f, T)^{-1} =$

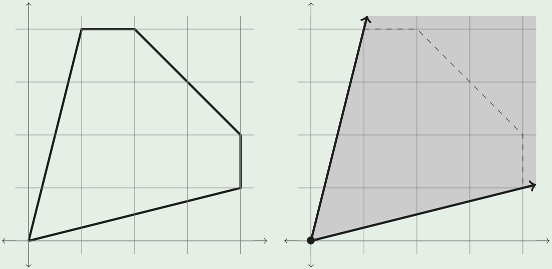
$$-27T^4 \quad + \quad 0T^3 \quad + \quad 18T^2 \quad + \quad 8T \quad + \quad 1$$

$$(4,3) \quad (3,\infty) \quad \overset{\downarrow}{(2,2)} \quad (1,0) \quad (0,0)$$

The Hodge Polygon

There exists a combinatorial lower bound to the Newton polygon called the Hodge polygon $HP(\Delta)$. This is constructed using the cone generated by Δ consisting of all rays passing through nonzero points of Δ emanating from the origin. This is denoted $C(\Delta)$.

Example



Hodge Polygon (continued)

For a vector u in $\mathbb{R}^n$, $w(u)$ is defined to be the smallest positive real number c such that $u \in c\Delta$. If no such c exists, that is, $u \notin C(\Delta)$, we define $w(u) = \infty$.
Let $D = D(\Delta)$. For an integer k, let

$$W_\Delta(k) = card\{u \in \mathbb{Z}^n | w(u) = \frac{k}{D}\}.$$

This is the number of lattice points in $\mathbb{Z}^n$ with weight k/D.

Hodge Polygon (continued)

Let

$$H_\Delta(k) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} W_\Delta(k - iD).$$

$H_\Delta(k)$ is a non-negative integer for each $k \geq 0$. Furthermore,

$$H_\Delta(k) = 0, \text{for } k > nD$$

and

$$\sum_{k=0}^{nD} H_\Delta(k) = n!\mathrm{Vol}(\Delta).$$

Hodge Polygon (continued)

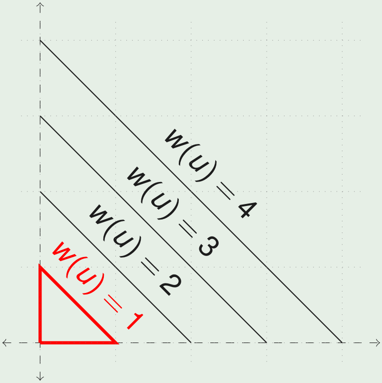
The Hodge polygon $HP(\Delta)$ of Δ is defined to be the lower convex polygon in $\mathbb{R}^2$ with vertices

$$\left(\sum_{k=0}^{m} H_\Delta(k), \frac{1}{D} \sum_{k=0}^{m} k H_\Delta(k) \right).$$

That is, the polygon $HP(\Delta)$ is the polygon starting from the origin with a side of slope k/D with horizontal length $H_\Delta(k)$ for each integer $0 \leq k \leq nD$.

Hodge Polygon (continued)

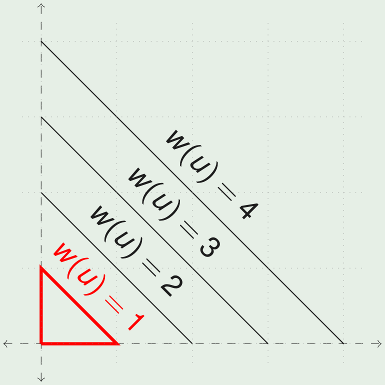
Example



For the polytope generated by $(0,0), (1,0)$ and $(0,1)$ we have $D = 1$.

Hodge Polygon (continued)

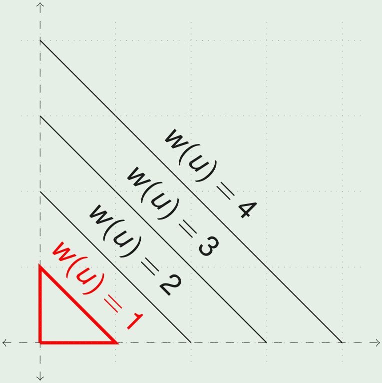
Example



From the diagram we see the first few $W_\Delta(k)$ are:
$W_\Delta(0) = 1$
$W_\Delta(1) = 2$
$W_\Delta(2) = 3$
$W_\Delta(3) = 4$
$W_\Delta(4) = 5.$

Hodge Polygon (continued)

Example

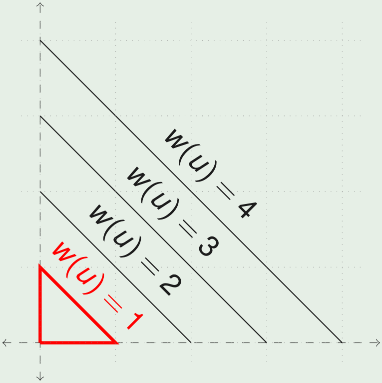


From this we get:

$HP_{\Delta}(0) = 1$
$HP_{\Delta}(1) = 0$
$HP_{\Delta}(2) = 0.$

Hodge Polygon (continued)

Example



Hence $HP(\Delta)$ is simply the horizontal line joining the origin and $(1, 0)$. This makes sense since $n!\mathrm{Vol}(\Delta) = 1$.

Main Question

Definition

When $NP(f) = HP(\Delta)$ we say f is **ordinary**.

Definition

Let $GNP(\Delta, p) = \inf_{f \in M_p(\Delta)} NP(f)$.

We know that $GNP(\Delta, p) \geq HP(\Delta)$ for every p.

Generic Ordinarity

Main Question

When is $GNP(\Delta, p) = HP(\Delta)$?

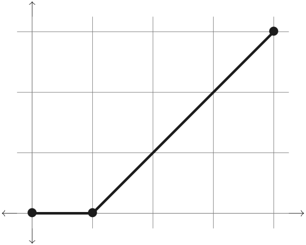
If $GNP(\Delta, p) = HP(\Delta)$ we say Δ is **generically ordinary** at p.

Conjecture

Adolphson and Sperber conjectured that if $p \equiv 1 \pmod{D(\Delta)}$ the $M_p(\Delta)$ is generically ordinary.

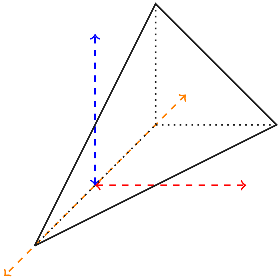
Wan showed that this is not quite true, but if we replace $D(\Delta)$ with an effectively computable $D^*(\Delta)$ this is true.

Big Example



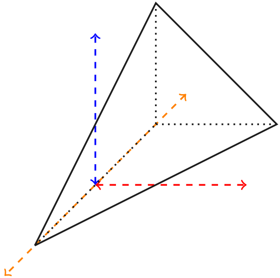
Recall for $p = q = 3$ and $f = \frac{1}{x_1} + x_1 x_2^2 + x_1 x_3^2$, the Newton polygon of $L(f, T)^{(-1)^{(n-1)}} = -27T^4 + 18T^2 + 8T + 1$. the Newton polygon is:

Big Example



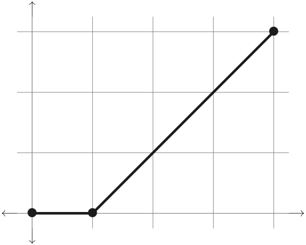
$\Delta(f)$ is the polytope spanned by the origin, (-1,0,0), $(1, 2, 0)$ and $(1, 0, 2)$.

Big Example



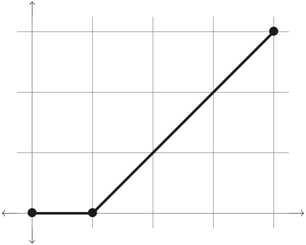
k	0	1	2	3
$W_\Delta(k)$	1	6	15	28
$H_\Delta(k)$	1	3	0	0

Big Example



m	0	1	2	3
$\sum_{k=0}^{m} H_\Delta(k)$	1	1	4	4
$\frac{1}{D}\sum_{k=0}^{m} kH_\Delta(k)$	0	0	3	3

Big Example



From this we see that the Newton Polygon is equal to the Hodge polygon. Hence f is ordinary.

Facial Decomposition

Let $\{\sigma_1, \ldots, \sigma_h\}$ be the set of faces of Δ that do not contain the origin.

Theorem (Facial Decomposition Theorem)

Let f be non-degenerate and let $\Delta(f)$ be n-dimensional. Then f is ordinary if and only if each f_{σ_i} is ordinary. Equivalently, f is non-ordinary if and only if if some f_{σ_i} is non-ordinary.

Using the facial decomposition theorem we may assume that $\Delta(f)$ is generated by a single codimension 1 face not containing the origin.

This allows us to concentrate on methods to decompose the individual faces of Δ.

Coherent Decomposition

Let δ be a face of Δ not containing the origin.

Definition

A **coherent** decomposition of δ is a decomposition T into polytopes $\delta_1, \ldots, \delta_h$ such that there is a piecewise linear function $\phi : \delta \mapsto \mathbb{R}$ such that

1. ϕ is concave i.e. $\phi(tx + (1-t)x') \geq t\phi(x) + (1-t)\phi(x')$, for all $x, x' \in \delta, 0 \leq t \leq 1$.
2. The domains of linearity of ϕ are precisely the n-dimensional simplices δ_i for $1 \leq i \leq m$.

Coherent decompositions are sometimes called concave decompositions.

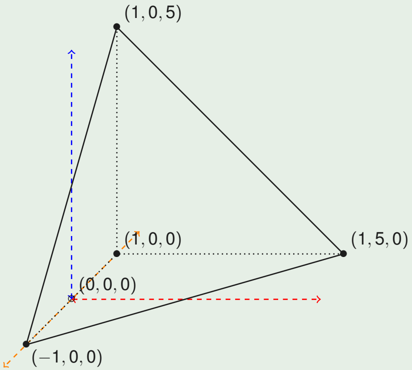
Coherent Decomposition Theorem

Let Δ be a polytope containing a unique face δ away from the origin. Let $\delta = \cup\delta_i$ be a complete coherent decomposition of δ. Let Δ_i denoted the convex closure of δ_i and the origin. Then $\Delta = \cup\Delta_i$. We call this a coherent decomposition of Δ.

Theorem (Coherent Decomposition (L-))

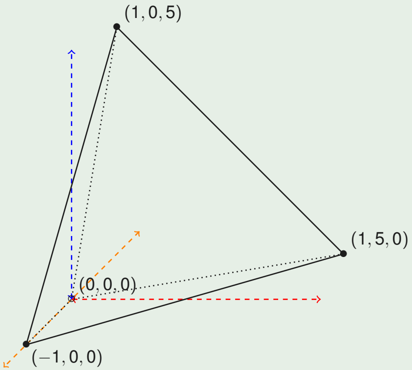
Suppose each lattice point of δ is a vertex of δ_i for some i. If each f_{Δ_i} is generically non-degenerate and ordinary for some prime p, then f is also generically non-degenerate and ordinary for the same prime p.

Example



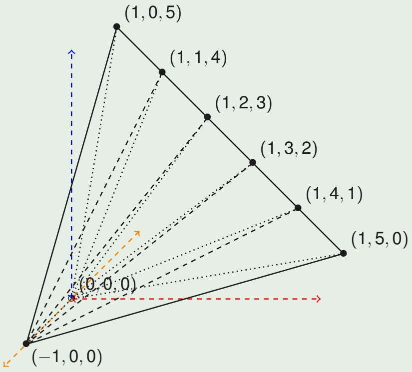
There are two faces away from the origin. Using the facial decomposition theorem we can divide this into two polytopes.

Example



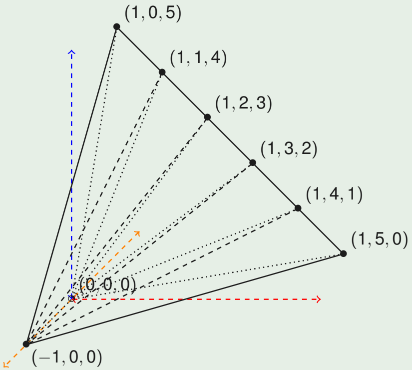
Consider the polytope Δ' with vertices $(0,0,0), (-1,0,0), (1,5,0)$ and $(1,0,5)$. Wan's work has shown that the back face is ordinary for any prime so we can ignore it.

Example



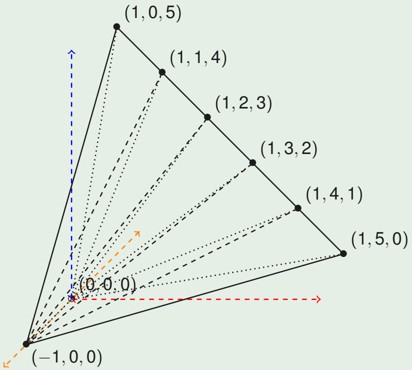
We can decompose the front face, which will decompose the entire polytope

Example



For any $f \in M_p(\Delta')$ if f is ordinary when restricted to each of these pieces, it is ordinary on all of f.

Example



One can show that $D(\Delta') = 5$ and Δ' is generically ordinary when $p \equiv 1 \pmod 5$, that is, Adolphson and Sperber's conjecture holds in this case.

Reducing $L(f, t)$

Using the Dwork trace formula one can reformulate $L(f, T)$:

$$L(f,T)^{(-1)^{(n-1)}} = \prod_{i=0}^{n} \det(I - Tq^i A_a(f))^{(-1)^{n-1}}$$

where $A_a(f)$ is an infinite Frobenius matrix.
Hence our understanding of $L(f, T)$ can be reduced to understanding $A_a(f)$.

$A_1(f)$

If we are just concerned with ordinarity and not $L(f, T)$ or the actual Newton polygon, we can focus on a much simpler function:

$$\det(I - TA_1(f))$$

where

$$A_a(f) = A_1 A_1^{\tau^1} \dots A_1^{\tau^{a-1}}$$

where τ is a lift of $x \mapsto x^p$ from $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ to a generator of $\mathrm{Gal}(K/\mathbb{Q}_p)$ for K is a degree a unramified extension of $\mathbb{Q}_p$.

Definition of $A_1(f)$

Let $\pi \in \overline{\mathbb{F}_p}$ satisfy $\sum_{m=0}^{\infty} \frac{\pi^{p^m}}{p^m} = 0$ with $\mathrm{ord}\pi = \frac{1}{p-1}$.Let

$$F_r(f) = \sum_u \left(\prod_{j=1}^{J} \lambda_{u_j} a_j^{u_j} \right) \pi^{u_1 + \ldots + u_J},$$

where the outer sum is over all solutions of the linear system

$$\sum_{j=1}^{J} u_j V_j = r, u_j \geq 0, u_j \text{ integral.}$$

For the purposes of Newton polygons we are mostly concerned with the $\pi^{u_1+\ldots+u_J}$ part.

Definition of $A_1(f)$ (continued)

$A_1(f)$ is the infinite matrix whose rows are indexed by r and columns are indexed by s, lattice points in the closed cone $C(\Delta)$:

$$A_1(f) = (a_{r,s}(f)) = (F_{ps-r}(f)\pi^{w(r)-w(s)}).$$

One can also derive the lower bound:

$$\mathrm{ord}\, a_{r,s}(f) \geq \frac{w(ps-r)+w(r)-w(s)}{p-1} \geq w(s).$$

Block form of $A_1(f)$

Let ξ be such that $\xi^D = \pi^{p-1}$. Hence $\mathrm{ord}\xi = 1/D$. By ordering r and s in terms of weights we can write

$$A_1(f) = \begin{pmatrix} A_{00} & \xi^1 A_{01} & \dots & \xi^i A_{0i} & \dots \\ A_{10} & \xi^1 A_{11} & \dots & \xi^i A_{1i} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ A_{i0} & \xi^1 A_{i1} & \dots & \xi^i A_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix},$$

where the block A_{ij} is a finite matrix with $W_\Delta(i)$ rows and $W_\Delta(j)$ columns.

Chain Polytope

- This block form of $A_1(f)$ give us a very stable foothold in understanding ordinarity.
- The ξ^i terms are precisely the parts that show $NP(f) \geq HP(\Delta)$.

Using this one can construct an $A_1(f)$ version of the Hodge polygon:

Definition

Let $P(\Delta)$ be the polygon in $\mathbb{R}^2$ with vertices $(0,0)$ and

$$\left(\sum_{k=0}^{m} W(\Sigma, k), \frac{1}{D(\Delta)} \sum_{k=0}^{m} kW(\Sigma, k)\right), \ m = 0, 1, 2, \ldots .$$

Chain Level

Theorem

The Newton polygon of $\det(I - TA_1(f))$ *is equal to* $P(\Delta)$ *if and only if* $NP(f) = HP(\Delta)$*, that is, when* f *is ordinary.*

- This allows us to examine $\det(I - TA_1(f))$ rather than the entire L-function. This is called working on the chain level.
- The main advantage of working on the chain level is the block representation of $A_1(f)$.

Cone Restriction of $A_1(f)$

Block Form

$$A_1(f) = (a_{r,s}(f)) = \begin{pmatrix} A_{00} & \xi^1 A_{01} & \dots & \xi^i A_{0i} & \dots \\ A_{10} & \xi^1 A_{11} & \dots & \xi^i A_{1i} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ A_{i0} & \xi^1 A_{i1} & \dots & \xi^i A_{ii} & \dots \\ \vdots & \vdots & \ddots & \vdots & \end{pmatrix}$$

Let $\Delta_1, \dots, \Delta_h$ be a coherent decomposition of Δ. Let $\Sigma_i = C(\Delta_i)$, the cone generated by Δ_i.
For a cone Σ, Let $A_1(\Sigma, f)$ be the "Σ" piece of $(a_{s,r}(f))$ in $A_1(f)$, that is, r and s run through the cone Σ rather than the entire cone $C(\Delta)$.

Cone Restriction of $P(\Delta)$

Since we have a cone restricted $A_1(f)$ we must also have a cone restricted $P(\Delta)$. Let

$$W(\Sigma, k) = \#\left\{r \in \mathbb{Z}^n \cap \Sigma \mid w(r) = \frac{k}{D(\Delta)}\right\}.$$

Definition

Let $P(\Sigma)$ be the polygon in $\mathbb{R}^2$ with vertices $(0,0)$ and

$$\left(\sum_{k=0}^{m} W(\Sigma, k), \frac{1}{D(\Delta)} \sum_{k=0}^{m} kW(\Sigma, k)\right), \quad m = 0, 1, 2, \ldots.$$

Outline of Proof

- The idea is to show that if Δ_i is a member of a coherent decomposition the entries in $A_1(f)$ with the highest order occur precisely when r and s are from the same cone $C(\Delta_i)$.
- Therefore these bad terms will also appear in $A_1(\Sigma_i, f)$ and we can compare it to $P(\Sigma_i)$ to determine ordinarity.
- One can show that considering $A_1(\Sigma_i, f)$ is equivalent to considering $A_1(\Sigma, f_{\Delta_i})$ for the purposes of ordinarity.
- Therefore if we assume each f_{Δ_i} is generically ordinary for all i, then the Newton polygon of $\det(I - TA_1(\Sigma_i, f))$ will coincide with $P(\Sigma_i)$ for all i. Then f itself will be chain level generically ordinary, which is equivalent to regular generic ordinary.

The End